

# Hamiltonian Formalism of Particular Bimetric Gravity Model

---

**J. Klusoň**

*Department of Theoretical Physics and Astrophysics  
Faculty of Science, Masaryk University  
Kotlářská 2, 611 37, Brno  
Czech Republic  
E-mail: klu@physics.muni.cz*

**ABSTRACT:** In this short note we perform the Hamiltonian analysis of bimetric gravity with one particular form of potential between two metrics. We find that this theory have eight secondary constraints. We identify four constraints that are the first class constraints on condition when the interaction term obeys some specific condition. We show that for the form of the potential studied in this paper this condition is obeyed and hence we can interpret these first class constraints as generators of the diagonal diffeomorphism.

**KEYWORDS:** Massive Gravity, Hamiltonian Formalism.

---

## Contents

<b>1. Introduction and Summary</b>	<b>1</b>
<b>2. Hamiltonian analysis of Bimetric gravity</b>	<b>2</b>

---

## 1. Introduction and Summary

Bimetric theories of gravity are basically defined as theories of two metrics  $\hat{g}_{\mu\nu}$  and  $\hat{f}_{\mu\nu}$  whose dynamics are governed by two Einstein-Hilbert actions together with the interaction term between  $\hat{g}_{\mu\nu}$  and  $\hat{f}_{\mu\nu}$ . The history of bimetric or more generally multimetric theories of gravity [4] is long, we recommend the paper [3] for list of the references to earlier works<sup>1</sup>. It was believed for the long time that these theories suffer from the presence of the ghost degree of freedom. However very recently the ghost free bimetric theory of gravity was suggested in [11]. The main novelty of this new bimetric theory of the gravity is the specific form of the interaction term between  $\hat{g}_{\mu\nu}$  and  $\hat{f}_{\mu\nu}$  that was firstly proposed in the formulation of the ghost-free non-linear massive theory of gravity [15, 16, 17, 18].

It is well known that the bimetric theories of gravity are invariant under diagonal diffeomorphism. More explicitly, without the interaction term the action for two bimetric theory of gravity is sum of two Einstein-Hilbert actions for  $\hat{g}_{\mu\nu}$  and  $\hat{f}_{\mu\nu}$  and each of these actions is diffeomorphism invariant. However the presence of the interaction term breaks this symmetry to the diagonal one. Since this is the gauge symmetry we should expect an existence of the four first class constraints in the Hamiltonian formalism of given theory. Remarkably it turns out that it is non-trivial task to find these constraints and to show that they are really the first class constraints. This short note is devoted to this analysis at least for same specific form of the potential term.

Explicitly, we perform the  $3 + 1$  splitting of the metric components  $\hat{g}_{\mu\nu}, \hat{f}_{\mu\nu}$  and determine corresponding Hamiltonian. Then, following seminar paper [3] we perform the redefinition of the lapse and shifts functions so that the Hamiltonian is linear in the new lapse function and the new shift functions. Clearly this is the essential condition for the existence of the diffeomorphism constraints however it is not sufficient. In fact, we have to show that these constraints are preserved during the time evolution of the system without imposing conditions on the Lagrange multipliers. In other words we have to show that the Poisson brackets between diffeomorphism constraints vanish on the constraint surface. We show that the right side of the Poisson bracket between Hamiltonian constraints is proportional to the particular linear combinations of the variation of the potential term and we show that this expression vanishes for the case of the potential studied in this paper. In

---

<sup>1</sup>For recent works, see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

other words we find that there exists the Hamiltonian constraint and three spatial diffeomorphism constraints and that the Poisson brackets between these constraints vanish on the constraint surface. We believe that this is non-trivial result since as far as we know such an analysis has not been performed yet. Then we also analyze the remaining constraints and we show that they are the second class constraints. Finally we determine the number of the physical degrees of freedom and we argue that they can be interpreted as massless graviton, massive graviton and one additional scalar mode at linearized level.

It is clear that our work has an important limitation in the special form of the potential that was chosen. The natural extension of this work is to extend given analysis to the case of the bimetric theories of gravity introduced in [11]. However even if it was argued there that these theories are ghost free it is not completely clear how to identify the generators of the diagonal diffeomorphism. In principle the analysis presented here could be applied to this case as well but it is not clear how to identify additional constraints in given theory that are crucial for the elimination of the scalar mode. This problem is currently under investigation.

## 2. Hamiltonian analysis of Bimetric gravity

We begin this section with the introduction of the bimetric theories of gravity. The basic idea of bimetric gravity is simple. We have two Einstein-Hilbert actions for two four dimensional metrics  $\hat{g}_{\mu\nu}, \hat{f}_{\mu\nu}$ ,  $\mu, \nu = 0, 1, 2, 3$  together with the interaction term that does not contain the derivatives of the metric <sup>2</sup>

$$S = M_L^2 \int d^4x \sqrt{-\hat{g}^{(4)}} R(\hat{g}) + M_R^2 \int d^4x \sqrt{-\hat{f}^{(4)}} R(\hat{f}) - \mu \int d^4x (\det \hat{g} \det \hat{f})^{1/4} \mathcal{V}(\hat{g}, \hat{f}) . \quad (2.1)$$

In this short note we consider following specific form of the potential  $\mathcal{V}$

$$\mathcal{V}(\hat{g}, \hat{f}) = \sum_n c_n (H^\mu{}_\mu)^n + \sum_m d_m (H^\mu{}_\nu H^\nu{}_\mu)^m , \quad (2.2)$$

where

$$H^\mu{}_\nu = \hat{g}^{\mu\rho} \hat{f}_{\rho\nu} , \quad (2.3)$$

and where  $c_n$  and  $d_n$  are numerical constants. Note that the action (2.1) is invariant under following diffeomorphism transformations

$$\hat{g}'_{\mu\nu}(x') = \hat{g}_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} , \quad \hat{f}'_{\mu\nu}(x') = \hat{f}_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} . \quad (2.4)$$

Our goal is to perform the Hamiltonian analysis of the theory defined by the action (2.1).

---

<sup>2</sup>We follow the notation used in [3].

To begin with we introduce the 3+1 decomposition of the four dimensional metric  $\hat{g}_{\mu\nu}$  [1, 2]

$$\begin{aligned}\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j , \quad \hat{g}_{0i} = N_i , \quad \hat{g}_{ij} = g_{ij} , \\ \hat{g}^{00} &= -\frac{1}{N^2} , \quad \hat{g}^{0i} = \frac{N^i}{N^2} , \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2}\end{aligned}\tag{2.5}$$

together with the metric  $\hat{f}_{\mu\nu}$

$$\begin{aligned}\hat{f}_{00} &= -M^2 + L_i f^{ij} L_j , \quad \hat{f}_{0i} = L_i , \quad \hat{f}_{ij} = f_{ij} , \\ \hat{f}^{00} &= -\frac{1}{M^2} , \quad \hat{f}^{0i} = \frac{L^i}{M^2} , \quad \hat{f}^{ij} = f^{ij} - \frac{L^i L^j}{M^2} , \quad L^i = L_j f^{ji} .\end{aligned}\tag{2.6}$$

Then using the well known relation <sup>3</sup>

$$\begin{aligned}{}^{(4)}R[\hat{g}] &= K_{ij} \mathcal{G}^{ijkl} K_{kl} + R^{(g)} , \\ {}^{(4)}R[\hat{f}] &= \tilde{K}_{ij} \tilde{\mathcal{G}}^{ijkl} \tilde{K}_{kl} + R^{(f)} ,\end{aligned}\tag{2.7}$$

where  $R^{(g)}$  and  $R^{(f)}$  are three dimensional scalar curvatures evaluated using the spatial metric  $g_{ij}$  and  $f_{ij}$  respectively and where the extrinsic curvatures  $K_{ij}$  and  $\tilde{K}_{ij}$  are defined as

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) , \quad \tilde{K}_{ij} = \frac{1}{2M}(\partial_t f_{ij} - \tilde{\nabla}_i L_j - \tilde{\nabla}_j L_i) ,\tag{2.8}$$

and where  $\nabla_i$  and  $\tilde{\nabla}_i$  are covariant derivatives evaluated using the metric components  $g_{ij}$  and  $f_{ij}$  respectively. Finally note that  $\mathcal{G}^{ijkl}$  and  $\tilde{\mathcal{G}}^{ijkl}$  are de Witt metrics defined as

$$\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl} , \quad \tilde{\mathcal{G}}^{ijkl} = \frac{1}{2}(f^{ik} f^{jl} + f^{il} f^{jk}) - f^{ij} f^{kl}\tag{2.9}$$

with inverse

$$\mathcal{G}_{ijkl} = \frac{1}{2}(g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{2}g_{ij} g_{kl} , \quad \tilde{\mathcal{G}}_{ijkl} = \frac{1}{2}(f_{ik} f_{jl} + f_{il} f_{jk}) - \frac{1}{2}f_{ij} f_{kl}\tag{2.10}$$

that obey the relation

$$\mathcal{G}_{ijkl} \mathcal{G}^{klmn} = \frac{1}{2}(\delta_i^m \delta_j^n + \delta_i^n \delta_j^m) , \quad \tilde{\mathcal{G}}_{ijkl} \tilde{\mathcal{G}}^{klmn} = \frac{1}{2}(\delta_i^m \delta_j^n + \delta_i^n \delta_j^m) .\tag{2.11}$$

Using (2.7) we rewrite the action (2.1) into the form that is suitable for the Hamiltonian analysis

$$\begin{aligned}S &= \int dt L = M_g^2 \int d^3 \mathbf{x} dt \sqrt{g} N [K_{ij} \mathcal{G}^{ijkl} K_{kl} + R^{(g)}] + \\ &+ M_f^2 \int d^3 \mathbf{x} dt M \sqrt{f} [\tilde{K}_{ij} \tilde{\mathcal{G}}^{ijkl} \tilde{K}_{kl} + R^{(f)}] - \mu \int d^3 \mathbf{x} dt g^{1/4} f^{1/4} \sqrt{NM} \mathcal{V} .\end{aligned}\tag{2.12}$$

---

<sup>3</sup>We ignore the boundary terms.

Then from (2.12) we determine following conjugate momenta

$$\begin{aligned}
\pi^{ij} &= \frac{\delta L}{\delta \partial_t g_{ij}} = M_g^2 \mathcal{G}^{ijkl} K_{kl} , \quad \rho^{ij} = \frac{\delta L}{\delta \partial_t f_{ij}} = M_f^2 \tilde{\mathcal{G}}^{ijkl} \tilde{K}_{kl} , \\
\pi_i &= \frac{\delta L}{\delta \partial_t N^i} \approx 0 , \quad \rho_i = \frac{\delta L}{\delta \partial_t L^i} \approx 0 , \\
\pi_N &= \frac{\delta L}{\delta \partial_t N} \approx 0 , \quad \rho_M = \frac{\delta L}{\delta \partial_t M} \approx 0
\end{aligned} \tag{2.13}$$

and then using the standard procedure we derive following Hamiltonian

$$H = \int d^3 \mathbf{x} (N \mathcal{R}_0^{(g)} + M \mathcal{R}_0^{(f)} + N^i \mathcal{R}_i^{(g)} + L^i \mathcal{R}_i^{(f)} + \mu \sqrt{NM} g^{1/4} f^{1/4} \mathcal{V}) , \tag{2.14}$$

where

$$\begin{aligned}
\mathcal{R}_0^{(g)} &= \frac{1}{M_g^2 \sqrt{g}} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - M_g^2 \sqrt{g} R^{(g)} , \quad \mathcal{R}_0^{(f)} = \frac{1}{M_f^2 \sqrt{f}} \rho^{ij} \tilde{\mathcal{G}}_{ijkl} \rho^{kl} - M_f^2 \sqrt{f} R^{(f)} , \\
\mathcal{R}_i^{(g)} &= -2g_{ij} \nabla_k \pi^{kj} , \quad \mathcal{R}_i^{(f)} = -2f_{ij} \tilde{\nabla}_k \rho^{kj} .
\end{aligned} \tag{2.15}$$

The crucial point is to identify four constraints that correspond to the diffeomorphism invariance of given theory. In order to do this we proceed as in [3] and introduce following variables

$$\begin{aligned}
\bar{N} &= \sqrt{NM} , \quad n = \sqrt{\frac{\bar{N}}{M}} , \quad \bar{N}^i = \frac{1}{2}(N^i + L^i) , \quad n^i = \frac{N^i - L^i}{\sqrt{NM}} , \\
N &= \bar{N} n , \quad M = \frac{\bar{N}}{n} , \quad L^i = \bar{N}^i - \frac{1}{2} n^i \bar{N} , \quad N^i = \bar{N}^i + \frac{1}{2} n^i \bar{N} ,
\end{aligned} \tag{2.16}$$

where again clearly their conjugate momenta are the primary constraints of the theory

$$P_{\bar{N}} \approx 0 , \quad P_n \approx 0 , \quad P_i \approx 0 , \quad p_i \approx 0 . \tag{2.17}$$

Note that the canonical variables have following non-zero Poisson brackets

$$\begin{aligned}
\{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} &= \frac{1}{2}(\delta_i^k \delta_j^l + \delta_j^k \delta_i^l) \delta(\mathbf{x} - \mathbf{y}) , \quad \{f_{ij}(\mathbf{x}), \rho^{kl}(\mathbf{y})\} = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_j^k \delta_i^l) \delta(\mathbf{x} - \mathbf{y}) , \\
\{\bar{N}(\mathbf{x}), P_{\bar{N}}(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}) , \quad \{n(\mathbf{x}), P_n(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) , \\
\{\bar{N}^i(\mathbf{x}), P_j(\mathbf{y})\} &= \delta_j^i \delta(\mathbf{x} - \mathbf{y}) , \quad \{n^i(\mathbf{x}), p_j(\mathbf{y})\} = \delta_j^i \delta(\mathbf{x} - \mathbf{y}) .
\end{aligned} \tag{2.18}$$

With the help of (2.16) we find the explicit form of the matrix  $H^\mu_\nu$

$$H^0_0 = \frac{1}{n^2} + \frac{n^i}{\bar{N} n^2} f_{ij} (\bar{N}^j - \frac{1}{2} \bar{N} n^j) , \quad H^0_j = \frac{1}{\bar{N} n^2} n^k f_{kj} ,$$

$$\begin{aligned}
H^i_0 &= -\frac{1}{n^4}(\bar{N}^i + \frac{1}{2}\bar{N}n^i) + g^{ij}f_{jk}(\bar{N}^k - \frac{1}{2}n^k\bar{N}) - \frac{1}{\bar{N}n^2}(\bar{N}^i + \frac{1}{2}n^i\bar{N})n^kf_{kl}(\bar{N}^l - \frac{1}{2}\bar{N}n^l) , \\
H^i_j &= g^{ik}f_{kj} - \frac{1}{\bar{N}n^2}(\bar{N}^i + \frac{1}{2}n^i\bar{N})n^kf_{kj}
\end{aligned} \tag{2.19}$$

so that

$$H^\mu_\mu = \frac{1}{n^4} - \frac{1}{n^2}n^if_{ij}f^j + g^{ij}f_{ji} \tag{2.20}$$

and also

$$H^\mu_\nu H^\nu_\mu = \frac{1}{n^8} - \frac{2}{n^6}n^if_{ij}n^j + \frac{1}{n^4}(n^if_{ij}n^j)^2 + g^{ij}f_{jk}g^{kl}f_{li} - \frac{2}{n^2}n^if_{ij}g^{jk}f_{kl}n^l . \tag{2.21}$$

As a result we find that the potential  $\mathcal{V}$  defined in (2.2) does not depend on  $\bar{N}, \bar{N}^i$  which is very important for the existence of the diffeomorphism constraints. Then using the variables (2.16) we find that the Hamiltonian takes the form

$$H = \int d^3\mathbf{x}(\bar{N}\mathcal{R} + \bar{N}^i\mathcal{R}_i) , \tag{2.22}$$

where

$$\begin{aligned}
\mathcal{R} &= n\mathcal{R}_0^{(g)} + \frac{1}{n}\mathcal{R}_0^{(f)} + \frac{1}{2}n^i\mathcal{R}_i^{(g)} - \frac{1}{2}n^i\mathcal{R}_i^{(f)} + \\
&+ \mu g^{1/4}f^{1/4}\mathcal{V} , \quad \mathcal{R}_i = \mathcal{R}_i^{(g)} + \mathcal{R}_i^{(f)} .
\end{aligned} \tag{2.23}$$

As usual the requirement of the preservation of the primary constraints (2.17) implies following secondary ones

$$\begin{aligned}
\partial_t P_{\bar{N}} &= \{P_{\bar{N}}, H\} = -\mathcal{R} \approx 0 , \\
\partial_t P_i &= \{P_i, H\} = -\mathcal{R}_i \approx 0 , \\
\partial_t P_n &= \{P_n, H\} = -\mathcal{R}_0^{(g)} + \frac{1}{n^2}\mathcal{R}_0^{(f)} - \mu g^{1/4}f^{1/4}\frac{\delta\mathcal{V}}{\delta n} \equiv \mathcal{G} \approx 0 , \\
\partial_t p_i &= \{p_i, H\} = -\frac{1}{2}\mathcal{R}_i^{(g)} + \frac{1}{2}\mathcal{R}_i^{(f)} - \mu g^{1/4}f^{1/4}\frac{\delta\mathcal{V}}{\delta n^i} \equiv \mathcal{G}_i \approx 0 .
\end{aligned} \tag{2.24}$$

For the consistency of the theory it is important to show that the constraints  $\mathcal{R}$  and  $\mathcal{R}_i$  are the first class constraints. To proceed it is useful to introduce the smeared form of the constraint  $\mathcal{R}$

$$\mathbf{T}_T(N) = \int d^3\mathbf{x}N(\mathbf{x})\mathcal{R}(\mathbf{x}) . \tag{2.25}$$

In case of the constraint  $\mathcal{R}_i$  it turns out that it is convenient to extend the constraint  $\mathcal{R}_i$  with appropriate combinations of the primary constraints  $P_n, p_i$  so that we define  $\tilde{\mathcal{R}}_i$  as

$$\tilde{\mathcal{R}}_i = \mathcal{R}_i^{(g)} + \mathcal{R}_i^{(f)} + \partial_inP_n + \partial_in^jp_j + \partial_j(n^jp_i) \tag{2.26}$$

and then define its smeared form

$$\mathbf{T}_S(N^i) = \int d^3\mathbf{x} N^i \tilde{\mathcal{R}}_i . \quad (2.27)$$

Finally it is useful to introduce the smeared forms of the constraints  $\mathcal{R}_0^{(f),(g)}$  and  $\mathcal{R}_i^{(f),(g)}$

$$\begin{aligned} \mathbf{T}_T^g(N) &= \int d^3\mathbf{x} N(\mathbf{x}) \mathcal{R}_0^{(g)}(\mathbf{x}) , & \mathbf{T}_T^f(N) &= \int d^3\mathbf{x} N(\mathbf{x}) \mathcal{R}_0^{(f)}(\mathbf{x}) , \\ \mathbf{T}_S^g(N^i) &= \int d^3\mathbf{x} N^i(\mathbf{x}) \mathcal{R}_i^{(g)}(\mathbf{x}) , & \mathbf{T}_S^f(N^i) &= \int d^3\mathbf{x} N^i(\mathbf{x}) \mathcal{R}_i^{(f)}(\mathbf{x}) . \end{aligned} \quad (2.28)$$

It is well known that these smeared constraints have following non-zero Poisson brackets <sup>4</sup>

$$\begin{aligned} \{\mathbf{T}_T^g(N), \mathbf{T}_T^g(M)\} &= \mathbf{T}_S^g((N\partial_i M - M\partial_i N)g^{ij}) , \\ \{\mathbf{T}_T^f(N), \mathbf{T}_T^f(M)\} &= \mathbf{T}_S^f((N\partial_i M - M\partial_i N)f^{ij}) , \\ \{\mathbf{T}_S^g(N^i), \mathbf{T}_T^g(M)\} &= \mathbf{T}_T^g(N^i\partial_i M) , \\ \{\mathbf{T}_S^f(N^i), \mathbf{T}_T^f(M)\} &= \mathbf{T}_T^f(N^i\partial_i M) , \\ \{\mathbf{T}_S^g(N^i), \mathbf{T}_S^g(M^j)\} &= \mathbf{T}_S^g((N^j\partial_j M^i - M^j\partial_j N^i)) , \\ \{\mathbf{T}_S^f(N^i), \mathbf{T}_S^f(M^j)\} &= \mathbf{T}_S^f((N^j\partial_j M^i - M^j\partial_j N^i)) . \end{aligned} \quad (2.29)$$

To proceed further note that using (2.18) and (2.27) we find

$$\begin{aligned} \{\mathbf{T}_S(N^i), g_{ij}\} &= -N^k \partial_k g_{ij} - \partial_i N^k g_{kj} - g_{ik} \partial_j N^k , \\ \{\mathbf{T}_S(N^i), \pi^{ij}\} &= -\partial_k (N^k \pi^{ij}) + \partial_k N^i \pi^{kj} + \pi^{ik} \partial_k N^j , \\ \{\mathbf{T}_S(N^i), f_{ij}\} &= -N^k \partial_k f_{ij} - \partial_i N^k f_{kj} - f_{ik} \partial_j N^k , \\ \{\mathbf{T}_S(N^i), \rho^{ij}\} &= -\partial_k (N^k \rho^{ij}) + \partial_k N^i \rho^{kj} + \rho^{ik} \partial_k N^j , \\ \{\mathbf{T}_S(N^i), n\} &= -N^i \partial_i n , \\ \{\mathbf{T}_S(N^i), P_n\} &= -\partial_i (N^i P_n) , \\ \{\mathbf{T}_S(N^i), n^i\} &= -N^k \partial_k n^i + \partial_j N^i n^j , \\ \{\mathbf{T}_S(N^i), p_i\} &= -\partial_k (N^k p_i) - \partial_i N^k p_k . \end{aligned} \quad (2.30)$$

From these results we see that  $\mathbf{T}_S(N)$  is the generator of the spatial diffeomorphism. Moreover, using previous Poisson brackets we easily find that

$$\{\mathbf{T}_S(N^i), \mathbf{T}_S(M^j)\} = \mathbf{T}_S((N^j\partial_j M^i - M^j\partial_j N^i)) . \quad (2.31)$$

---

<sup>4</sup>See, for example [20].

On the other hand more interesting is to determine the Poisson bracket between smeared forms of the Hamiltonian constraints (2.25)

$$\begin{aligned}
\{\mathbf{T}_T(N), \mathbf{T}_T(M)\} &= \{\mathbf{T}_T^g(nN), \mathbf{T}_T^g(nM)\} + \left\{ \mathbf{T}_T^f\left(\frac{1}{n}N\right), \mathbf{T}_T^f\left(\frac{1}{n}M\right) \right\} + \\
&+ \left\{ \mathbf{T}_T^g(Nn), \mathbf{T}_S^g\left(\frac{1}{2}Mn^i\right) \right\} + \left\{ \mathbf{T}_S^g\left(\frac{1}{2}Nn^i\right), \mathbf{T}_T^g(Mn) \right\} + \\
&+ \left\{ \mathbf{T}_T^g(nN), \int d^3\mathbf{x} N \mu g^{1/4} f^{1/4} \mathcal{V} \right\} + \left\{ \int d^3\mathbf{x} M \mu g^{1/4} f^{1/4} \mathcal{V}, \mathbf{T}_T^g(nM) \right\} + \\
&- \left\{ \mathbf{T}_T^f\left(N\frac{1}{n}\right), \mathbf{T}_S^f\left(\frac{1}{2}Mn^i\right) \right\} - \left\{ \mathbf{T}_S^f\left(\frac{1}{2}Nn^i\right), \mathbf{T}_T^f\left(M\frac{1}{n}\right) \right\} + \\
&+ \left\{ \mathbf{T}_T^f\left(\frac{1}{n}N\right), \int d^3\mathbf{x} N \mu g^{1/4} f^{1/4} \mathcal{V} \right\} + \left\{ \int d^3\mathbf{x} M \mu g^{1/4} f^{1/4} \mathcal{V}, \mathbf{T}_T^f\left(\frac{1}{n}M\right) \right\} + \\
&+ \frac{1}{4} \left\{ \mathbf{T}_S^g(Nn^i), \mathbf{T}_S^g(Mn^j) \right\} + \frac{1}{4} \left\{ \mathbf{T}_S^f(Nn^i), \mathbf{T}_S^f(Mn^j) \right\} + \\
&+ \frac{1}{2} \left\{ \mathbf{T}_S^g(Nn^i), \int d^3\mathbf{x} M \mu f^{1/4} g^{1/4} \mathcal{V} \right\} + \frac{1}{2} \left\{ \int d^3\mathbf{x} \mu N f^{1/4} g^{1/4} \mathcal{V}, \mathbf{T}_S^g(Mn^i) \right\} - \\
&- \frac{1}{2} \left\{ \mathbf{T}_S^f(Nn^i), \int d^3\mathbf{x} M \mu f^{1/4} g^{1/4} \mathcal{V} \right\} - \frac{1}{2} \left\{ \int d^3\mathbf{x} \mu N f^{1/4} g^{1/4} \mathcal{V}, \mathbf{T}_S^f(Mn^i) \right\} = \\
&= \mathbf{T}_S((N\partial_i M - M\partial_i N)n^2 g^{ij}) + \mathbf{T}_S((N\partial_i M - M\partial_i N)\frac{1}{n^2} f^{ij}) - \\
&- \mathbf{G}_S((N\partial_i M - M\partial_i N)n^2 g^{ij}) - \mathbf{G}_T((N\partial_i M - M\partial_i N)n^i) + \\
&+ \mathbf{G}_S((N\partial_i M - M\partial_i N)\frac{1}{n^2} f^{ij}) + \int d^3\mathbf{x} (N\partial_i M - M\partial_i N) \Sigma^i,
\end{aligned} \tag{2.32}$$

where we defined the smeared forms of the constraints  $\mathcal{G}_i$  and  $\mathcal{G}$

$$\mathbf{G}_T(N) = \int d^3\mathbf{x} N(\mathbf{x}) \mathcal{G}(\mathbf{x}), \quad \mathbf{G}_S(N^i) = \int d^3\mathbf{x} N^i(\mathbf{x}) \mathcal{G}_i(\mathbf{x}), \tag{2.33}$$

and where  $\Sigma^i$  is defined as

$$\Sigma^i = \mu g^{1/4} f^{1/4} \left[ -n^2 g^{ij} \frac{\delta \mathcal{V}}{\delta n^j} + f^{ij} \frac{\delta \mathcal{V}}{\delta n^j} \frac{1}{n^2} - \frac{\delta \mathcal{V}}{\delta g^{kj}} n^k g^{ij} - \frac{\delta \mathcal{V}}{\delta f_{ij}} n^k f_{kj} - \frac{1}{2} n^i n \frac{\delta \mathcal{V}}{\delta n} \right]. \tag{2.34}$$

We see that the Poisson bracket between the smeared forms of the Hamiltonian constraints (2.32) vanish on the constraint surface on condition that  $\Sigma_i$  is zero. We explicitly check below that this is indeed the case for the potential (2.2). In fact, using

$$\begin{aligned}
\frac{\delta H_\mu^\mu}{\delta n} &= -\frac{4}{n^5} + \frac{2n^i f_{ij} n^j}{n^3}, & \frac{\delta H_\mu^\mu}{\delta n^i} &= -2 \frac{f_{ij} n^j}{n^2}, \\
\frac{\delta H_\mu^\mu}{\delta g^{ij}} &= f_{ij}, & \frac{\delta H_\mu^\mu}{\delta f_{ij}} &= g^{ij} - \frac{n^i n^j}{n^2}
\end{aligned} \tag{2.35}$$



and

$$\begin{aligned}
\frac{\delta(H_\nu^\mu H_\mu^\nu)}{\delta n} &= -\frac{8}{n^9} + 12\frac{n^i f_{ij} n^j}{n^7} - \frac{4}{n^5}(n^i f_{ij} n^j)^2 + \frac{4}{n^3}n^m f_{mi} g^{ij} f_{jk} n^k, \\
\frac{\delta(H_\nu^\mu H_\mu^\nu)}{\delta n^i} &= -4\frac{f_{ij} n^j}{n^6} + \frac{4}{n^4}f_{ij} n^j (n^k f_{kl} n^l) - \frac{4}{n^2}f_{ij} g^{jm} f_{mn} n^n, \\
\frac{\delta(H_\nu^\mu H_\mu^\nu)}{\delta f_{ij}} &= -2\frac{n^i n^j}{n^6} + \frac{2}{n^4}n^i n^j (n^k f_{kl} n^l) + 2g^{im} f_{mn} g^{nj} - \frac{4}{n^2}g^{im} f_{mn} n^n n^j, \\
\frac{\delta(H_\nu^\mu H_\mu^\nu)}{\delta g^{ij}} &= 2f_{im} g^{mn} f_{nj} - \frac{2}{n^2}n^m f_{mi} f_{jn} n^n
\end{aligned} \tag{2.36}$$

and after some tedious calculations we find that  $\Sigma^i = 0$  for the potential (2.2). In summary, we derived the fundamental result that the Poisson bracket between Hamiltonian constraint (2.32) vanishes on the constraint surface.

As the next step we determine the Poisson bracket between  $\mathbf{T}_S(N^i)$  and  $\mathbf{T}_T(N)$ . We firstly determine following Poisson bracket

$$\begin{aligned}
\left\{ \mathbf{T}_S(N), g^{1/4} f^{1/4} \mathcal{V} \right\} &= -N^k \partial_k [g^{1/4} f^{1/4} \mathcal{V}] - \partial_k N^k g^{1/4} f^{1/4} \mathcal{V} + \\
&+ g^{1/4} f^{1/4} \left[ \frac{\delta \mathcal{V}}{\delta n^i} \partial_j N^i n^j - 2 \frac{\delta \mathcal{V}}{\delta f_{kl}} \partial_k N^m f_{ml} + 2 \frac{\delta \mathcal{V}}{\delta g^{kl}} \partial_m N^k g^{ml} \right] = \\
&= -N^k \partial_k [g^{1/4} f^{1/4} \mathcal{V}] - \partial_k N^k g^{1/4} f^{1/4} \mathcal{V}
\end{aligned} \tag{2.37}$$

where in the final step we used (2.35) and (2.36). Then with the help of (2.29) and (2.30) we obtain

$$\begin{aligned}
\left\{ \mathbf{T}_S(N^i), \mathbf{T}_T(M) \right\} &= \mathbf{T}_T^g(N^i \partial_i M n) + \mathbf{T}_T^f(N^i \partial_i M \frac{1}{n^2}) + \\
&+ \frac{1}{2} \mathbf{T}_S^g(N^j \partial_j M n^i) - \frac{1}{2} \mathbf{T}_S^f(N^j \partial_j M n^i) + \\
&+ \int d^3 \mathbf{x} N^k \partial_k M \mu g^{1/4} f^{1/4} \mathcal{V} = \mathbf{T}_T(N^i \partial_i M).
\end{aligned} \tag{2.38}$$

This result together with (2.31) and (2.32) shows that  $\mathbf{T}_T(N)$  and  $\mathbf{T}_S(N^i)$  are the first class constraints that are generators of the diagonal diffeomorphism.

Now we proceed to the analysis of constraints  $\mathcal{G}_i, \mathcal{G}_n$ . For further purposes we introduce following "Hamiltonian"

$$\mathbf{H}(\bar{N}, \bar{N}^i) = \mathbf{T}_T(\bar{N}) + \mathbf{T}_S(\bar{N}^i) \tag{2.39}$$

so that the total Hamiltonian has the form

$$H_T = \mathbf{H}(\bar{N}, \bar{N}^i) + \int d^3 \mathbf{x} (v^n P_n + v^i p_i + u^n \mathcal{G}_n + u^i \mathcal{G}_i). \tag{2.40}$$

Then the requirement of the preservation of the primary constraints  $P_n, p_i$  implies

$$\begin{aligned}
\partial_t P_n &= \{P_n, H_T\} = \mathcal{G} + \int d^3\mathbf{x} (u^n(\mathbf{x}) \{P_n, \mathcal{G}_n(\mathbf{x})\} + u^i \{P_n, \mathcal{G}_i(\mathbf{x})\}) \approx \\
&\approx \int d^3\mathbf{x} (u^n(\mathbf{x}) \{P_n, \mathcal{G}_n(\mathbf{x})\} + u^i \{P_n, \mathcal{G}_i(\mathbf{x})\}) = 0 \\
\partial_t p_i &= \{p_i, H_T\} = \mathcal{G}_i + \int d^3\mathbf{x} (u^n(\mathbf{x}) \{p_i, \mathcal{G}_n(\mathbf{x})\} + u^j \{p_i, \mathcal{G}_j(\mathbf{x})\}) \approx 0 \\
&\approx \int d^3\mathbf{x} (u^n(\mathbf{x}) \{p_i, \mathcal{G}_n(\mathbf{x})\} + u^j \{p_i, \mathcal{G}_j(\mathbf{x})\}) = 0 .
\end{aligned} \tag{2.41}$$

We have four equations for four unknown  $u_n$  and  $u_i$  that can be solved for  $u_n, u_i$  at least in principle. Further, the requirement of the preservation of the constraints  $\mathcal{G}_n, \mathcal{G}_i$  gives next four equations for unknown  $v_n$  and  $v_i$  that can again be explicitly solved<sup>5</sup>. In other words  $P_n, p_i, \mathcal{G}_n, \mathcal{G}_i$  are the second class constraints.

For example, let us consider the simplest form of the potential

$$\mathcal{V} = H^\mu_\mu . \tag{2.42}$$

In this case we find

$$\begin{aligned}
\mathcal{G}_n &= -\mathcal{R}_0^{(g)} + \frac{1}{n^2} \mathcal{R}^{(f)} + \mu f^{1/4} g^{1/4} \left( -\frac{4}{n^5} + 2 \frac{n^i f_{ij} n^j}{n^3} \right) , \\
\mathcal{G}_i &= -\frac{1}{2} \mathcal{R}_i^{(g)} + \frac{1}{2} \mathcal{R}_i^{(f)} + 2\mu \frac{g^{1/4} f^{1/4}}{n^2} f_{ij} n^j
\end{aligned} \tag{2.43}$$

and hence

$$\begin{aligned}
\{P_n(\mathbf{x}), \mathcal{G}_n(\mathbf{y})\} &= \left( \frac{2}{n^3} \mathcal{R}_0^{(f)} - 20 f^{1/4} g^{1/4} \frac{\mu}{n^6} + 6\mu f^{1/4} g^{1/4} \frac{n^i f_{ij} n^j}{n^4} \right) \delta(\mathbf{x} - \mathbf{y}) \\
&\equiv \Delta_{nn}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) , \\
\{P_n(\mathbf{x}), \mathcal{G}_i(\mathbf{y})\} &= \frac{4\mu}{n^4} g^{1/4} f^{1/4} f_{ij} n^j(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \equiv \Delta_{ni}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) , \\
\{p_i(\mathbf{x}), \mathcal{G}_n(\mathbf{y})\} &= -4\mu f^{1/4} g^{1/4} \frac{f_{ij} n^j}{n^3}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \equiv \Delta_{in}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) , \\
\{p_i(\mathbf{x}), \mathcal{G}_j(\mathbf{y})\} &= -2\mu g^{1/4} f^{1/4} \frac{f_{ij}}{n^2}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \equiv \Delta_{ij}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) .
\end{aligned} \tag{2.44}$$

Then the first equation in (2.41) can be solved for  $u_n$  as

$$u^n = -\frac{\Delta_{ni} u^i}{\Delta_{nn}} , \tag{2.45}$$

---

<sup>5</sup>However we should stress that there is a possibility that with the suitable form of the potential there could exist additional constraints. Such a form of the potential is well known in the case of the non-linear massive gravity case and corresponding bi-metric generalization [4, 11].

Note that  $\Delta_{nn}$  is non-zero for the generic point of the phase space. Inserting this result into the second equation in (2.41) we obtain the homogeneous equation for  $u^i$

$$\left(\Delta_{ij} - \frac{\Delta_{in}\Delta_{nj}}{\Delta_{nn}}\right)u^j = \Delta_{im}\left(\delta_j^m - (\Delta^{-1})^{mk}\frac{\Delta_{kn}\Delta_{nj}}{\Delta_{nn}}\right)u^j \equiv \Delta_{im}M_j^m u^j = 0. \quad (2.46)$$

Now it is easy to see that the matrix  $M_j^i = \delta_j^i - \frac{8\mu g^{1/4}f^{1/4}}{n^5\Delta_{nn}}n^i n^k f_k$  is non-singular at the generic points of the phase space so that the only solution of the equation (2.46) is given by  $u^i = 0$  and consequently  $u^n = 0$  as follows from (2.45). Then it is easy to analyze the time evolution of the constraints  $\mathcal{G}_n, \mathcal{G}_i$

$$\begin{aligned} \partial_t \mathcal{G}_n &= \{\mathcal{G}_n, H_T\} = \{\mathcal{G}_n, \mathbf{H}(\bar{N}, \bar{N}^i)\} + \\ &+ \int d^3\mathbf{x} (v^i(\mathbf{x}) \{\mathcal{G}_n, p_i(\mathbf{x})\} + v^n(\mathbf{x}) \{\mathcal{G}_n, P_n(\mathbf{x})\}) = 0, \\ \partial_t \mathcal{G}_i &= \{\mathcal{G}_i, H_T\} = \{\mathcal{G}_i, \mathbf{H}(\bar{N}, \bar{N}^i)\} + \\ &+ \int d^3\mathbf{x} (v^j(\mathbf{x}) \{\mathcal{G}_i, p_j(\mathbf{x})\} + v^n(\mathbf{x}) \{\mathcal{G}_i, P_n(\mathbf{x})\}) = 0 \end{aligned} \quad (2.47)$$

which are four equations for four unknown functions  $v^n, v^i$ . Then using the same arguments as in case of the constraints  $P_n, p_i$  we find that these equations can be solved for  $v^n, v^i$  as functions of the canonical variables and the Lagrange multipliers  $\bar{N}, \bar{N}^i$ .

Let us outline the nature of various constraints and the number of the physical degrees of freedom in the theory. We have following eight second class constraints:  $P_n \approx 0, p_i \approx 0, \mathcal{G}_n \approx 0, \mathcal{G}_i \approx 0$ . Solving these constraints we find that  $P_n, p_i$  vanish strongly and solving  $\mathcal{G}_n = 0, \mathcal{G}_i = 0$  we can express  $n, n_i$  as functions of remaining canonical variables. We also have four first class constraints  $P_{\bar{N}} \approx 0, P_i \approx 0$ . Gauge fixing of these constraints we can eliminate  $P_{\bar{N}}, P_i$  together with  $\bar{N}, \bar{N}^i$ . Finally we have 24 phase space variables  $g_{ij}, \pi^{ij}, f_{ij}, \rho^{ij}$  together with four first class constraints  $\mathcal{R} \approx 0, \mathcal{R}_i \approx 0$ . Then using the standard counting of the physical degrees of freedom we find that the number of the phase space degrees of freedom is 16 where four of them correspond to the massless graviton while 10 of them can be interpreted as the massive graviton. However there are two additional degrees of freedom corresponding to the scalar mode. Clearly such a mode cannot be eliminated for the generic point of the potential  $\mathcal{V}$ . On the other hand as we stressed in the introduction there are examples of the suitable chosen potentials that lead to the potentially ghost free bimetric or multimetric theories of gravity [4, 11]. It is natural step to extend the analysis presented in this work to this case as well.

**Acknowledgement:** This work was supported by the Grant agency of the Czech republic under the grant P201/12/G028.

## References

- [1] E.ourgoulhon, “3+1 formalism and bases of numerical relativity,” gr-qc/0703035 [GR-QC].
- [2] R. L. Arnowitt, S. Deser, C. W. Misner, “The Dynamics of general relativity,” [gr-qc/0405109].

- [3] T. Damour and I. I. Kogan, “*Effective Lagrangians and universality classes of nonlinear bigravity*,” Phys. Rev. D **66** (2002) 104024 [hep-th/0206042].
- [4] K. Hinterbichler and R. A. Rosen, “*Interacting Spin-2 Fields*,” JHEP **1207** (2012) 047 [arXiv:1203.5783 [hep-th]].
- [5] F. Kuhnel, “*On Instability of Certain Bi-Metric and Massive-Gravity Theories*,” arXiv:1208.1764 [gr-qc].
- [6] S. F. Hassan, A. Schmidt-May and M. von Strauss, “*On Consistent Theories of Massive Spin-2 Fields Coupled to Gravity*,” arXiv:1208.1515 [hep-th].
- [7] C. Deffayet, J. Mourad and G. Zahariade, “*Covariant constraints in ghost free massive gravity*,” arXiv:1207.6338 [hep-th].
- [8] K. Nomura and J. Soda, “*When is Multimetric Gravity Ghost-free?*,” Phys. Rev. D **86** (2012) 084052 [arXiv:1207.3637 [hep-th]].
- [9] S. F. Hassan, A. Schmidt-May and M. von Strauss, “*Metric Formulation of Ghost-Free Multivielbein Theory*,” arXiv:1204.5202 [hep-th].
- [10] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell and S. F. Hassan, “*Cosmological Solutions in Bimetric Gravity and their Observational Tests*,” JCAP **1203** (2012) 042 [arXiv:1111.1655 [gr-qc]].
- [11] S. F. Hassan and R. A. Rosen, “*Bimetric Gravity from Ghost-free Massive Gravity*,” JHEP **1202** (2012) 126 [arXiv:1109.3515 [hep-th]].
- [12] M. Banados, A. Gomberoff and M. Pino, “*The bigravity black hole and its thermodynamics*,” Phys. Rev. D **84** (2011) 104028 [arXiv:1105.1172 [gr-qc]].
- [13] M. Banados and S. Theisen, “*Three-dimensional massive gravity and the bigravity black hole*,” JHEP **0911** (2009) 033 [arXiv:0909.1163 [hep-th]].
- [14] M. Banados, A. Gomberoff, D. C. Rodrigues and C. Skordis, “*A Note on bigravity and dark matter*,” Phys. Rev. D **79** (2009) 063515 [arXiv:0811.1270 [gr-qc]].
- [15] C. de Rham, G. Gabadadze and A. J. Tolley, “*Resummation of Massive Gravity*,” Phys. Rev. Lett. **106** (2011) 231101 [arXiv:1011.1232 [hep-th]].
- [16] S. F. Hassan and R. A. Rosen, “*On Non-Linear Actions for Massive Gravity*,” JHEP **1107** (2011) 009 [arXiv:1103.6055 [hep-th]].
- [17] S. F. Hassan and R. A. Rosen, “*Resolving the Ghost Problem in non-Linear Massive Gravity*,” Phys. Rev. Lett. **108** (2012) 041101 [arXiv:1106.3344 [hep-th]].
- [18] C. de Rham, G. Gabadadze and A. J. Tolley, “*Ghost free Massive Gravity in the Stückelberg language*,” Phys. Lett. B **711** (2012) 190 [arXiv:1107.3820 [hep-th]].
- [19] S. F. Hassan, R. A. Rosen and A. Schmidt-May, “*Ghost-free Massive Gravity with a General Reference Metric*,” JHEP **1202** (2012) 026 [arXiv:1109.3230 [hep-th]].
- [20] S. A. Hojman, K. Kuchar and C. Teitelboim, “*Geometrodynamics Regained*,” Annals Phys. **96** (1976) 88.